Journal of Statistical Physics, Vol. 91, Nos. 1/2, 1998

Scaling Limits of Solutions of the Heat Equation for Singular Non-Gaussian Data

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Received December 31, 1996

Limiting distributions of the parabolically rescaled solutions of the heat equation with singular non-Gaussian initial data with long-range dependence are described in terms of their multiple stochastic integral representations.

KEY WORDS: Heat equation; scaling limit; non-Gaussian initial conditions; Chebyshev–Hermite expansions; long-range dependence.

1. INTRODUCTION

The heat equation with random initial conditions is a classical subject that has been extensively studied in both mathematical and physical literature. An introduction of the rigorous probabilistic tools into the subject can be traced to Kampé de Fériet (1955) and Rosenblatt (1968) who considered the heat equation with stationary initial conditions and gave the spectral representation of stationary solutions in the form of stochastic integrals.

More recently, several researchers investigated solutions of the heat equation depending on various types of random initial conditions, including the nonhomogeneous case with random potentials (see, for example, Becus (1980), Nualart and Zakai (1989), Rotanov *et al.* (1991), Carmona and Molchanov (1995), Ubøe and Zhang (1995), Holden *et al.* (1996), Noble (1997), and the references therein). In particular, Rotanov *et al.* (1991) studied the convergence of the statistical solutions of the heat equation (and more general linear parabolic equations) with weakly dependent

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0022-4715/98/0400-0423$15.00/0

initial conditions to the Gaussian limiting process. Carmona and Molchanov (1995) and Noble (1997) present the asymptotic formulae for moments of solutions of the heat equation with random potential and study the intermittency phenomenon.

In the present paper we study the limiting processes of parabolically rescaled solutions of the heat equation emerging in the case when the initial condition is a nonlinear transformation of a stationary Gaussian process with long-range dependence. In a sense, our results are analogous to the limit theorems for nonlinear functionals of Gaussian processes and fields with long-range dependence (see, for example, Dobrushin and Major (1979), Taqqu (1975, 1979), Ivanov and Leonenko (1989), Leonenko and Olenko (1992) and others), but the type of non-Gaussian limiting processes obtained in this paper is new.

On the other hand, paradoxically, it was our work on the more complex random initial-value problem for nonlinear diffusion equations (see, e.g., Bulinski and Molchanov (1991), Albeverio *et al.* (1994), Funaki *et al.* (1995), Hu and Woyczynski (1994), Leonenko and Orsingher (1995), Molchanov *et al.* (1995), Surgailis and Woyczynski (1994)) that spurred us to reexamine the seemingly well studied situation for the classical heat equation.

2. THE MAIN RESULT

We consider the classical one-dimensional heat equation

$$\frac{\partial u}{\partial t}=\mu\frac{\partial^2 u}{\partial x^2}, \qquad t>0, \quad x\in\mathbf{R}^1, \quad \mu>0 \tag{2.1}$$

subject to the random initial condition

$$u(0, x)=\eta(x) \tag{2.2}$$

where $u=u(t, x)$. The process $\eta(x)=\eta(x;\omega)$, $x\in\mathbf{R}^1$ (defined on a suitable complete probability space $(\Omega, \mathscr{F}, P)$) is assumed to be a measurable, mean-square continuous, wide-sense stationary stochastic process with the expectation

$$m=E\eta(x)$$

and the covariance function

$$R(x)=\operatorname{cov}(\eta(0), \eta(x)), \qquad x\in\mathbf{R}^1$$

The Bochner–Khinchin Theorem assures that the covariance function $R(x)$ has the spectral representation

$$R(x) = \int_{-\infty}^{\infty} e^{i\lambda x} F(d\lambda)$$

where the spectral measure F is bounded and positive on $(\mathbf{R}^1, \mathscr{B}(\mathbf{R}^1))$. In view of Karhunen's Theorem, there exists a complex-valued orthogonally scattered random measure $Z(\cdot)$ such that, for every $x \in \mathbf{R}^1$, process $\eta(x)$ itself has the spectral representation (P-a.s.)

$$\eta(x) = m + \int_{-\infty}^{\infty} e^{i\lambda x} Z(d\lambda)$$

where

$$E\,|Z(\Delta)|^2 = F(\Delta), \qquad \Delta \in \mathscr{B}(\mathbf{R}^1)$$

and the stochastic integral is viewed as an L^2 integral with the control measure $F(\cdot)$.

The solution $u(t, x)$, $t > 0$, $x \in \mathbf{R}^1$, of the initial-value problem (2.1)–(2.2) can be written as the usual convolution

$$u(t, x) = \int_{-\infty}^{\infty} \eta(y) \frac{\exp\{-(x-y)^2/(4\mu t)\}}{\sqrt{4\pi\mu t}}\, dy \tag{2.2a}$$

of the initial data with the Gaussian kernel. Using the Karhunen's representation we can immediately see that the solution field $u(t, x)$ is a stationary process in x with the (P-a.s.) spectral representation (see, Rosenblatt (1968), for the case of discrete control measure F)

$$u(t, x) = m + \int_{-\infty}^{\infty} e^{i\lambda x - \mu\lambda^2 t} Z(d\lambda)$$

If $\eta(x)$, $x \in \mathbf{R}^1$, is a stationary Gaussian process with spectral density $f(\lambda)$, $\lambda \in \mathbf{R}^1$, then $u(t, x)$, $t > 0$, $x \in \mathbf{R}^1$, is a stationary in x Gaussian field with covariance structure

$$\operatorname{cov}(u(t, x), u(t', x')) = \int_{-\infty}^{\infty} e^{i\lambda(x-x')} f(\lambda)\, e^{2\lambda^2(t+t')}\, d\lambda$$

If the process $\eta(x)$, $x \in \mathbf{R}^1$, is subordinate to a Gaussian process (see, Dobrushin (1979)), then $u(t, x)$, $t > 0$, $x \in \mathbf{R}^1$, can be written as a series of Wiener–Itô integrals with corresponding transfer functions. These transfer functions express the non-Gaussian structure of the field $u(t, x)$, $t>0$, $x \in \mathbf{R}^1$. In particular, it is possible to calculate the spectral densities of higher order using the diagram formalism of Dobrushin (1979). These expansions are also useful in simulations of the random field $u(t, x)$ (see, for example, Nualart and Zakai (1989) and Holden *et al.* (1996), for an exposition of the Wiener–Itô expansions for the heat equation with random additive potential, and Kwapien and Woyczynski (1992) as a general source on multiple Wiener–Itô integrals).

A large open area of investigation is to consider the rescaled solutions of the heat equation with random initial conditions and random potential. In this paper, we shall restrict ourselves to finding the limiting distributions of the parabolically rescaled solutions of the initial-value problem (2.1–2.2) in the case, where the stochastic process $\eta(x)$, $x \in \mathbf{R}^1$, is a pointwise transformation of a stationary Gaussian process $\xi(x)$, $x \in \mathbf{R}^1$, i.e.,

$$\eta(x) = G(\xi(x)), \qquad x \in \mathbf{R}^1 \tag{2.3}$$

where the non-random function $G: \mathbf{R}^1 \to \mathbf{R}^1$ is such that $EG^2(\xi(0)) < \infty$. Our result indicates, in particular, that with non-Gaussian initial data with strong dependence the limiting distribution of the rescaled solutions may be non-Gaussian.

The underlying stationary process $\xi(x)$, $x \in \mathbf{R}^1$, is assumed to satisfy the following two conditions:

Condition A. The process $\xi(x)$ is a real, measurable, mean-square continuous stationary Gaussian process with mean $E\xi(x) = 0$ and variance $E\xi^2(x) = 1$, and correlation function

$$B(|x|) = E\xi(0)\,\xi(x) = \frac{L(|x|)}{|x|^{\alpha}}, \qquad 0 < \alpha < 1, \quad x \in \mathbf{R}^1$$

where $L(t)$, $t>0$, is a slowly varying function for large values of t and bounded on each finite interval, i.e., $L: (0, \infty) \to (0, \infty)$ and, for each $\lambda > 0$, $\lim_{t \to \infty}[L(\lambda t)/L(t)] = 1$.

Condition B. The spectral density $f(\lambda)$, $\lambda \in \mathbf{R}^1$, of the process $\xi(x)$ exists and is a decreasing function of λ for $\lambda \geqslant \lambda_0 > 0$.

Under conditions A and B, the correlation function $B(|x|)$ has the spectral representation

$$B(|x|) = 2\int_0^{\infty} \cos(\lambda x)\, f(\lambda)\, d\lambda \tag{2.4}$$

and, in view of the Tauberian theorem (see, Bingham *et al.* (1987), Leonenko and Olenko (1991)),

$$f(\lambda) \sim \alpha\lambda^{\alpha-1} \frac{L(1/\lambda)}{2c_1(\alpha)}, \qquad \lambda \to 0_+ \tag{2.5}$$

where

$$c_1(\alpha) = \frac{\Gamma(1+\alpha/2)\, 2^\alpha \sqrt{\pi}}{\Gamma((1-\alpha)/2)} = \Gamma(1+\alpha) \cos\frac{\alpha\pi}{2}, \qquad 0<\alpha<1$$

The process $\xi(x)$ itself has the spectral representation

$$\xi(x) = \int_{-\infty}^{\infty} e^{i\lambda x} \sqrt{f(\lambda)}\, W(d\lambda) \tag{2.6}$$

where $W(\cdot)$ is the complex Gaussian white noise (see, e.g., Major (1981), Kwapien and Woyczynski (1992)).

The nonlinear function $G(u)$, already assumed to satisfy condition $EG^2(\xi(x)) < \infty$ (see (2.3)), may be expanded in the series

$$G(u) = \sum_{k=0}^{\infty} C_k H_k(u)/k!, \qquad C_k = \int_{-\infty}^{\infty} G(u)\, \varphi(u)\, H_k(u)\, du \tag{2.7}$$

of orthogonal Chebyshev–Hermite polynomials

$$H_m(u) = (-1)^m e^{u^2/2} \frac{d^m}{du^m} e^{-u^2/2}, \qquad m = 0, 1, \ldots$$

which form a complete orthogonal basis in the Hibert space $L^2(\mathbf{R}^1, \varphi(u)\, du)$, with

$$\varphi(w) \equiv (2\pi)^{-1/2} \exp\{-w^2/2\}$$

Additionally, we will assume that function G satisfies

Condition C. There exists an integer $m \geqslant 1$ such that

$$C_1 = \cdots = C_{m-1} = 0, \qquad C_m \neq 0$$

The integer m will be called the Hermitian rank of G (see, e.g., Dobrushin and Major (1979), Taqqu (1979)).

Our basic description of the scaling limit distribution of the solution $u(t, x)$ is provided in terms of multiple stochastic integrals (see, for example, Kwapien and Woyczynski (1992)).

Theorem 2.1. Let $u(t, x)$, $t>0$, $x \in \mathbf{R}^1$, be a solution of the initial value problem (2.1-2) with the random initial data $\eta(x)=G(\xi(x))$ satisfying the above listed conditions A–C with $\alpha \in (0, 1/m)$, where $m \geqslant 1$ is the Hermite rank of function $G(u)$. Then, the finite dimensional distributions of the random field

$$X_T(t, x)=\frac{T^{\alpha m/4}}{L^{m/2}(\sqrt{T})}\left[u(tT, x\sqrt{T})-C_0\right]$$

converge weakly, as $T \to \infty$, to the finite-dimensional distributions of the random field $Z_m(t, x)$, with the following spectral multiple stochastic integral representation:

$$Z_m(t, x)=\left[\frac{\alpha}{2c_1(\alpha)}\right]^{m/2} \frac{C_m}{m!} \int'_{\mathbf{R}^m} \frac{e^{ix(\lambda_1+\cdots+\lambda_m)-\mu t(\lambda_1+\cdots+\lambda_m)^2}}{|\lambda_1 \cdot \cdots \cdot \lambda_m|^{(1-\alpha)/2}} \times W(d\lambda_1)\cdots W(d\lambda_m) \tag{2.8}$$

where C_k, $k=0, 1, 2,...$, are defined by (2.7), the constant $c_1(\alpha)$ is defined by (2.5), and the multiple stochastic integral $\int'$ is taken with the respect to the complex Gaussian white noise W over $\mathbf{R}^m$ with diagonal hyperplanes $\lambda_i=\pm\lambda_j$, $i, j=1,..., m$, $i \neq j$, excluded.

For any $m \geqslant 1$, and $0<\alpha<1/m$, the random field $Z_m(t, x)$, $x \in \mathbf{R}^1$, $t>0$, is stationary in x with expectation $EZ_m(t, x)=0$, and correlation function

$$\begin{aligned} EZ_m(t, x)\, Z_m(t', x') &= \left[\frac{\alpha}{2c_1(\alpha)}\right]^m \frac{C_m^2}{m!} \int_{\mathbf{R}^m} \frac{e^{i(x-x')(\lambda_1+\cdots+\lambda_m)-\mu(t+t')(\lambda_1+\cdots+\lambda_m)^2}}{|\lambda_1 \cdot \cdots \cdot \lambda_m|^{1-\alpha}}\, d\lambda_1 \cdots d\lambda_m \\ &= \frac{C_m^2}{m!}(2\mu)^{-m\alpha/2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\varphi(w_1)\,\varphi(w_2)\, dw_1\, dw_2}{|(x-x')/\sqrt{2\mu}-(w_1\sqrt{t}-w_2\sqrt{t'})|^{m\alpha}} \end{aligned} \tag{2.9}$$

A proof of the above result will be provided in the next section. Observe that the process $Z_1(t, x)$, $x \in \mathbf{R}^1$, $t>0$, is a Gaussian field which is stationary in x, with zero mean and the spectral density

$$g(\lambda)=e^{-(t+t')\mu\lambda^2}\,|\lambda|^{(\alpha-1)}\,[C_1^2\alpha/(2c_1(\alpha))], \qquad \lambda \in \mathbf{R}^1$$

such that

$$EZ_m(t, x)\, Z_m(t', x') = \int_{-\infty}^{\infty} e^{i\lambda(x-x')} g(\lambda)\, d\lambda$$

Remark 2.1. The above theorem should be compared with results of Taqqu (1975, 1979) and Dobrushin and Major (1979), who proved that, under conditions A, B, and C, the finite dimensional distributions of the stochastic process

$$Y_T(a) = \frac{1}{T^{1-\alpha m/2} L^{m/2}(T)} \int_0^{Ta} G(\xi(x))\, dx, \qquad 0 \leqslant a \leqslant 1$$

converge weakly to the finite-dimensional distributions of the stochastic process

$$Y_m(a) = \left(\frac{\alpha}{2c_1(\alpha)}\right)^{m/2} \int'_{\mathbf{R}^m} \frac{e^{i(\lambda_1 + \cdots + \lambda_m)a} - 1}{i(\lambda_1 + \cdots + \lambda_m)} \times \frac{W(d\lambda_1) \cdots W(d\lambda_m)}{|\lambda_1 \cdots \lambda_m|^{(1-\alpha)/2}}, \qquad 0 < \alpha < 1/m \tag{2.10}$$

Process $Y_1(a)$ is usually called the fractional Brownian motion, and process $Y_2(a)$—the Rosenblatt process (see, Rosenblatt (1961), Taqqu (1975)). For a process with continuous parameter the proof of this result may be obtained by using an argument from Berman (1979). On the other hand, using (2.5), it is easy to prove that $\lim_{T\to\infty} E\,|Y_T(a) - Y_m(a)|^2 = 0$ and then apply the Cramer–Wold arguments. Observe, however, that processes $Y_m(a)$ and $Z_m(1, x))$ are different, and in particular $Y_m(a)$ is not stationary.

Remark 2.2. The case $G(u) = e^{-u}$, $u \in \mathbf{R}^1$, was considered by Bulinskii and Molchanov (1991), and Albeverio *et al.* (1994); then, the Hermitian range $m = 1$.

3. PROOF OF THEOREM 2.1

Let $L^2(\Omega)$ be the Hilbert space of random variables with finite second moments. In view of (2.2a) and (2.7), the solution $u(t, x)$ has the Hermite expansion

$$\begin{aligned} u(t, x) - C_0 &= \int_{-\infty}^{\infty} G(\xi(y))\, \varphi_t(x-y)\, dy - C_0 \\ &= \sum_{k=m}^{\infty} \frac{C_k}{k!} \int_{-\infty}^{\infty} H_k(\xi(y))\, \varphi_t(x-y)\, dy \end{aligned} \tag{3.1}$$

where

$$\varphi_t(x) \equiv \frac{e^{-x^2/(4\mu t)}}{\sqrt{4\pi\mu t}}$$

Then, from (3.1),

$$u(tT, x\sqrt{T}) - C_0 = \frac{C_m}{m!} \zeta_{m,T}(t, x) + \sum_{k=m+1}^{\infty} \frac{C_k}{k!} \zeta_{k,T}(t, x) + V_m(T) \tag{3.2}$$

where

$$\zeta_{k,T}(t, x) = \int_{-T}^{T} H_k(\xi(y))\, \varphi_{tT}(x\sqrt{T} - y)\, dy, \qquad k = m, m+1, \ldots$$

and

$$V_m(T) = \sum_{k=m}^{\infty} \frac{C_k}{k!} \int_{|y| > T} H_k(\xi(y))\, \varphi_{tT}(x\sqrt{T} - y)\, dy$$

It is well known (see, for example, Ivanov and Leonenko (1989), p. 55) that

$$EH_k(\xi(y_1))\, H_j(\xi(y_2)) = k!\, \delta_k^j \{B(|y_1 - y_2|)\}^k, \qquad k, j \geqslant 0 \tag{3.3}$$

where δ_k^j is the usual Kronecker symbol.

From (3.3), we have

$$E\zeta_{k,T}(t, a)\, \zeta_{j,T}(t', b) = \delta_k^j E\zeta_{k,T}(t, a)\, \zeta_{k,T}(t', b), \qquad k, j \geqslant 1 \tag{3.4}$$

where

$$E\zeta_{k,T}(t, a)\, \zeta_{k,T}(t', b)$$
$$= k! \int_{-T}^{T} \int_{-T}^{T} \varphi_{tT}(a\sqrt{T} - y_1)\, \varphi_{t'T}(b\sqrt{T} - y_2)\, B^k(|y_1 - y_2|)\, dy_1\, dy_2$$

The last integral is easier to analyze after the change of variables

$$\frac{w_1^2}{2} = \frac{(a\sqrt{T} - y_1)^2}{4\mu t T}, \qquad \frac{w_2^2}{2} = \frac{(b\sqrt{T} - y_2)^2}{4\mu t T}$$

Then

$$E\zeta_{k,T}(t,a)\,\zeta_{k,T}(t',b)$$
$$=k!\int_{A_T(t,a)}\int_{A_T(t',b)}\varphi(w_1)\,\varphi(w_2)$$
$$\times B^k(|\sqrt{T}\,[(a-b)-(w_1\sqrt{t}-w_2\sqrt{t'})\sqrt{2\mu}\,])|)\,dw_1\,dw_2$$

where

$$A_T(t,a)=\left[\frac{a}{\sqrt{2\mu t}}-\sqrt{\frac{T}{2\mu t}},\frac{a}{\sqrt{2\mu t}}+\sqrt{\frac{T}{2\mu t}}\right]$$

Bearing in mind the properties of slowly varying functions (see, for example, Ivanov and Leonenko (1989), p. 56) we have, for $0<\alpha<1/k$, $k=m, m+1,\ldots,$ and $T\to\infty$,

$$E\zeta_{k,T}(t,a)\,\zeta_{k,T}(t',b)$$
$$=\frac{k!\,L^k(\sqrt{T})}{(2\mu T)^{k\alpha/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\varphi(w_1)\,\varphi(w_2)\,dw_1\,dw_2}{|(a-b)/\sqrt{2\mu}-(w_1\sqrt{t}-w_2\sqrt{t'})|^{k\alpha}}(1+o(1)) \tag{3.5}$$

It is easy to see that

$$\frac{\operatorname{var}\zeta_{k,T}(t,a)}{k!}\leqslant\frac{\operatorname{var}\zeta_{r,T}(t,a)}{r!}\qquad\text{if}\quad r\leqslant k$$

so that, from (3.4), we get

$$\operatorname{var}\left[\sum_{k=m+1}^{\infty}\frac{C_k}{k!}\zeta_{k,T}(t,a)\right]=\sum_{k=m+1}^{\infty}\frac{C_k^2}{(k!)^2}\operatorname{var}\zeta_{k,T}(t,a)$$
$$\leqslant\frac{\operatorname{var}\zeta_{m+1,T}(t,a)}{(m+1)!}\sum_{k=m+1}^{\infty}\frac{C_k^2}{k!} \tag{3.6}$$

Since we assumed that the Condition A is satisfied, for any $\varepsilon>0$ there exists an $U>0$ such that $B(|y_1-y_2|)<\varepsilon$ for $|y_1-y_2|>U$.

Now, let

$$\Delta_1=\{(w_1,w_2):|w_1-w_2|\leqslant U/\sqrt{2\mu tT},\ w_i\in A_T(t,a),\ i=1,2\}$$
$$\Delta_2=\{(w_1,w_2):|w_1-w_2|>U/\sqrt{2\mu tT},\ w_i\in A_T(t,a),\ i=1,2\}$$

and let us analyze the variance $\operatorname{var} \zeta_{m+1, T}(t, a)$ by splitting it into two parts as follows:

$$\begin{aligned}&\operatorname{var} \zeta_{m+1, T}(t, a)\\&=\left[\iint_{\Delta_1}+\iint_{\Delta_2}\right] \varphi(w_1)\, \varphi(w_2)\, B^{m+1}(|w_1-w_2| \sqrt{2\mu t T})\, dw_1\, dw_2\\&=S_1+S_2\end{aligned}$$

It is easy to see that

$$S_1 \leqslant K_1 \exp\{-s_T^2(t, a)\}, \qquad s_T(t, a)=\sqrt{T/2\mu t}-a/\sqrt{2\mu t}$$
$$S_2 \leqslant K_2 \varepsilon L(\sqrt{3\mu t T})\, T^{-\alpha m/2}$$

where $K_1, H_2>0$ are constants.

So, we have

$$\operatorname{var} \zeta_{m+1, T}(t, a) \leqslant K_1 e^{-s_T^2(t, a)}+K_2 \varepsilon L(\sqrt{2\mu t T})\, T^{-\alpha m/2} \tag{3.7}$$

In view of (3.7) and (3.5) (in the case $k=m$) the ratio

$$\frac{\operatorname{var} \zeta_{m+1, T}(t, a)}{\operatorname{var} \zeta_{m, T}(t, a)} \rightarrow 0, \qquad T \rightarrow \infty \tag{3.8}$$

since $\varepsilon>0$ is arbitrarily small.

From (3.6), (3.8), and Chebyshev's inequality we also get that

$$\frac{\sum_{k=m+1}^{\infty} C_k \zeta_{k, T}(t, a)/k!}{[\operatorname{var} \zeta_{m, T}(t, a)]^{1/2}} \xrightarrow{P} 0, \qquad T \rightarrow \infty \tag{3.9}$$

It is easy to see that for the remainder term in (3.2)

$$\operatorname{var} V_m(T) \leqslant K_3 \varphi(s_T(t, a)) \tag{3.10}$$

where $K_3>0$ is a constant.

From (3.10), we have

$$V_m(T)/[\operatorname{var} \zeta_{m, T}(t, a)]^{1/2} \xrightarrow{P} 0, \qquad T \rightarrow \infty \tag{3.11}$$

From (3.2), (3.9) and (3.11) we obtain

$$u(tT, x\sqrt{T})-C_0=\frac{C_m}{m!} \zeta_{m, T}(t, a)+v_T \tag{3.12}$$

where

$$v_T//[\operatorname{var} \zeta_{m,T}(t, a)]^{1/2} \xrightarrow{P} 0, \qquad T \to \infty$$

From (3.5) and (3.12) we have that the limiting distributions, as $T \to \infty$, of random variables

$$X_T(t, x) = \frac{T^{\alpha m/4}}{L^{m/2}(\sqrt{T})} [u(tT, x\sqrt{T}) - C_o], \qquad a \in \mathbf{R}^1$$

and random variables

$$\tilde{X}_{m,T}(t, x) = \frac{T^{\alpha m/4}}{L^{m/2}(\sqrt{T})} \frac{C_m}{m!} \zeta_{m,T}(t, x), \qquad x \in \mathbf{R}^1 \tag{3.13}$$

coincide, i.e., if the limiting distribution of one collection of random variables exists, then so does the limiting distribution of the other, and they are equal.

We shall prove that, for $0 < \alpha < 1/m$,

$$\lim_{T \to \infty} E\{\tilde{X}_{m,T}(t, x) - Z_m(t, x)\}^2 = 0 \tag{3.14}$$

where $\tilde{X}_{m,T}(t, x)$, $t > 0$, $x \in \mathbf{R}^1$, is defined by (3.13) and $Z_m(t, x)$, $t > 0$, $x \in \mathbf{R}^1$, is defined by (2.8).

Using Itô's formula (see, e.g., Dobrushin, Major (1979), Major (1981)), we obtain from (2.6) that

$$H_m(\xi(y)) = \int'_{\mathbf{R}^m} e^{iy(\lambda_1 + \cdots + \lambda_m)} \prod_{j=1}^{m} \sqrt{f(\lambda_j)}\, W(d\lambda_1) \cdots W(d\lambda_m) \tag{3.15}$$

Using the scaling property $W(d(a\lambda)) =^d \sqrt{a}\, W(d\lambda)$ and the well known formula

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\mu t}} \exp\left\{-\frac{(y-a)^2}{4\mu t} - iy(\lambda_1 + \cdots + \lambda_m)\right\} dy$$
$$= \exp\{i(\lambda_1 + \cdots + \lambda_m)\, a - \mu t(\lambda_1^2 + \cdots + \lambda_m^2)\}, \qquad \mu > 0, \quad a \in \mathbf{R}^1$$

we obtain from (3.13) and (3.15) after transformations: $y/\sqrt{T} = y'$ and $\sqrt{T}\, \lambda_j = \lambda'_j$, $j = 1, \ldots, m$, the following expression

$$\tilde{X}_{m,T}(t,x)$$

$$= \frac{T^{\alpha m/4}}{L^{m/2}(\sqrt{T})} \frac{C_m}{m!} \int'_{\mathbf{R}^m} \left[\int_{-T}^{T} e^{i(\lambda_1 + \cdots + \lambda_m) y} \varphi_{tT}(x\sqrt{T} - y)\, dy \right]$$

$$\times \prod_{j=1}^{m} \sqrt{f(\lambda_j)}\, W(d\lambda_1) \cdots W(d\lambda_m)$$

$$= \frac{T^{\alpha m/4}}{L^{m/2}(\sqrt{T})} \frac{C_m}{m!} \int'_{\mathbf{R}^m} \left[\int_{-\sqrt{T}}^{\sqrt{T}} e^{i(\lambda_1 + \cdots + \lambda_m) z\sqrt{T}} \frac{e^{-(z-x)^2/(4\mu t)}}{\sqrt{4\pi\mu t}}\, dz \right]$$

$$\times \prod_{j=1}^{m} \sqrt{f(\lambda_j)}\, W(d\lambda_1) \cdots W(d\lambda_m)$$

$$\stackrel{d}{=} \frac{T^{\alpha m/4}}{L^{m/2}(\sqrt{T})} \frac{C_m}{m!} \int'_{\mathbf{R}^m} \left[\int_{-\sqrt{T}}^{\sqrt{T}} e^{i(\lambda'_1 + \cdots + \lambda'_m) z\sqrt{T}} \frac{e^{-(z-x)^2/(4\mu t)}}{\sqrt{4\pi\mu t}}\, dz \right]$$

$$\times \frac{1}{T^{m/4}} \prod_{j=1}^{m} \sqrt{f(\lambda'_j/\sqrt{T})}\, W(d\lambda'_1) \cdots W(d\lambda'_m)$$

$$\stackrel{d}{=} \frac{C_m}{m!} \int'_{\mathbf{R}^m} \left[\int_{-\infty}^{\infty} e^{i(\lambda_1 + \cdots + \lambda_m) z} \frac{e^{-(z-x)^2/(4\mu t)}}{\sqrt{4\pi\mu t}}\, dz \right.$$

$$\left. - \int_{|z| > \sqrt{T}} e^{i(\lambda_1 + \cdots + \lambda_m) z} \frac{e^{-(z-x)^2/(4\mu t)}}{\sqrt{4\pi\mu t}}\, dz \right]$$

$$\times \frac{T^{(\alpha-1) m/4}}{L^{m/2}(\sqrt{T})} \prod_{j=1}^{m} \sqrt{f(\lambda_j/\sqrt{T})}\, W(d\lambda_1) \cdots W(d\lambda_m)$$

$$= \frac{C_m}{m!} \int'_{\mathbf{R}^m} e^{i(\lambda_1 + \cdots + \lambda_m) x - (\lambda_1 + \cdots + \lambda_m)^2 \mu t} \frac{T^{(\alpha-1) m/4}}{L^{m/2}(\sqrt{T})}$$

$$\times \prod_{j=1}^{m} \sqrt{f(\lambda_j/\sqrt{T})}\, W(d\lambda_1) \cdots W(d\lambda_m)$$

$$- \frac{C_m}{m!} \int'_{\mathbf{R}^m} \left[\int_{|z| > \sqrt{T}} e^{i(\lambda_1 + \cdots + \lambda_m) z} \frac{e^{-(z-x)^2/(4\mu t)}}{\sqrt{4\pi\mu t}}\, dz \right]$$

$$\times \frac{T^{(\alpha-1) m/4}}{L^{m/2}(\sqrt{T})} \prod_{j=1}^{m} \sqrt{f(\lambda_j/\sqrt{T})}\, W(d\lambda_1) \cdots W(d\lambda_m)$$

$$= \tilde{X}^{(1)}_{m,T}(t,x) - \tilde{X}^{(2)}_{m,T}(t,x) \tag{3.16}$$

From (3.16), we obtain

$$E\{\tilde{X}_{m,T}(t,x)-Z_m(t,x)\}^2 \leqslant 2\{E\,|\tilde{X}^{(1)}_{m,T}(t,x)-Z_m(t,x)|^2+E\,|X^{(2)}_{m,T}(t,x)|^2\} \tag{3.17}$$

We note that

$$\begin{aligned} &E\,|\tilde{X}^{(1)}_{m,T}(t,x)-Z_m(t,x)|^2 \\ &\quad=\frac{C_m^2}{(m!)^2}E\left|\int'_{\mathbf{R}^m} e^{i(\lambda_1+\cdots+\lambda_m)x-(\lambda_1+\cdots+\lambda_m)^2\mu t}\right. \\ &\qquad\times\left[\frac{T^{(\alpha-1)m/4}}{L^{m/2}(\sqrt{T})}\prod_{j=1}^{m}\sqrt{f(\lambda_j/\sqrt{T})}\right. \\ &\qquad\left.\left.-\left(\frac{\alpha}{2c_1(\alpha)}\right)^{m/2}\frac{1}{|\lambda\cdots\lambda_m|^{(1-\alpha)/2}}\right]W(d\lambda_1)\cdots W(d\lambda_m)\right|^2 \\ &\quad=\frac{C_m^2}{(m!)^2}\int'_{\mathbf{R}^m}\left|\frac{e^{i(\lambda_1+\cdots+\lambda_m)x-(\lambda_1+\cdots+\lambda_m)^2\mu t}}{|\lambda_1\cdots\lambda_m|^{(1-\alpha)/2}}\right|^2 Q_T(\lambda_1,\dots,\lambda_m)\,d\lambda_1\cdots d\lambda_m \end{aligned} \tag{3.18}$$

where

$$Q_T(\lambda_1,\dots,\lambda_m)=\left[\frac{|\lambda_1\cdots\lambda_m|^{(1-\alpha)/2}\,T^{\alpha m/4}}{T^{m/4}L^{m/2}(\sqrt{T})}\prod_{j=1}^{m}\sqrt{f(\lambda_j/\sqrt{T})}-\left(\frac{\alpha}{c_1(\alpha)}\right)^{m/2}\right]^2$$

Applying (2.7), we have

$$Q_T(\lambda_1,\dots,\lambda_m)\sim\left(\frac{\alpha}{2c_1(\alpha)}\right)^m\left[\prod_{j=1}^{m}\frac{L^{1/2}(\sqrt{T}/\lambda_j)}{L^{1/2}(\sqrt{T})}-1\right]^2,\qquad T\to\infty \tag{3.19}$$

The function

$$g(\lambda_1,\dots,\lambda_m)=\left|\frac{e^{i(\lambda_1+\cdots+\lambda_m)x-(\lambda_1+\cdots+\lambda_m)^2\mu t}}{|\lambda\cdots\lambda_m|^{(1-\alpha)/2}}\right|^2$$

is absolutely integrable. Using the properties of slowly varying functions (see, for example, Bingham et al. (1987) or Ivanov and Leonenko (1989), p. 56) we have, from (3.18–3.19), that

$$\lim_{t\to\infty}E\,|\tilde{X}^{(1)}_{m,T}(t,x)-Z_m(t,x)|^2=0 \tag{3.20}$$

The proof that

$$\lim_{T \to \infty} E\,|\tilde{X}^{(2)}_{m,T}(t,x)|^2 = 0 \tag{3.21}$$

follows from the trivial inequality

$$\lim_{T \to \infty} E\,|\tilde{X}^{(2)}_{m,T}(t,x)|^2$$

$$\leqslant \lim_{t \to \infty} \frac{C_m^2}{m!} \frac{T^{\alpha m/2}}{T^{m/2} L^m(\sqrt{T})} \int_{\mathbf{R}^m} \prod_{j=1}^m f\left(\frac{\lambda_j}{\sqrt{T}}\right)$$

$$\times \left| \int_{-\infty}^{\infty} \frac{e^{i(\lambda_1 + \cdots + \lambda_m)x - (z-x)^2/4\mu t}}{\sqrt{4\pi\mu t}}\, dz \right|^2 d\lambda_1 \cdots d\lambda_m = 0$$

From (3.20) and (3.12) we obtain (3.14). So

$$\tilde{X}_{m,T}(t,x) \xrightarrow{d} Z_m(t,x), \qquad t>0, \quad x \in \mathbf{R}^1$$

as $T \to \infty$. Applying the Cramer–Wold arguments we conclude the statement of the theorem. The formula (2.10) follows from (2.9) and (3.5). ∎

Remark 3.1. A small modification of arguments of Breuer and Major (1983), Bulinskii and Molchanov (1991), Surgailis and Woyczynski (1994), Albeverio *et al.* (1994), and Leonenko and Deriev (1994) also gives the following result describing the scaling limit solutions of the heat equation with weakly dependent (possibly non-Gaussian) initial conditions.

Let $\xi(x)$, $x \in \mathbf{R}^1$, be a stationary Gaussian process with $E\xi(x) = m$, $E\xi^2(x) = 1$, and covariance function $B(x)$, $x \in \mathbf{R}^1$. Consider the solution (2.2a) of the heat equation with $\eta(x)$ given by (2.3) where the non-random function G has the Hermitian rank $m \geqslant 1$ (see condition C). Suppose that

$$\int_{-\infty}^{\infty} |B(x)|^m\, dx < \infty$$

Then, the finite-dimensional distributions of the random fields

$$U_T(t,x) = T^{1/4}(u(tT, x\sqrt{T}) - C_0), \qquad t>0, \quad x \in \mathbf{R}^1$$

$$C_0 = \int_{-\infty}^{\infty} G(u)\, \varphi(u)\, du$$

converge weakly to the finite-dimensional distributions of a stationary in x Gaussian random field $U(t, x)$, $t > 0$, $x \in \mathbf{R}^1$, with $EU(t, x) = 0$ and the covariance function of the form

$$EU(t, x)\, T(t', x') = \frac{p \exp\{-|x - x'|^2/4\mu(t + t')\}}{\sqrt{4\pi\mu(t + t')}}$$

where

$$p = \int_{-\infty}^{\infty} \left[\sum_{k=m}^{\infty} \frac{C_k^2}{k!} B^k(x) \right] dx$$

Note that the corresponding spectral density has the form

$$g(\lambda) = \frac{p^2}{4\pi\mu(t + t')} \exp\left\{ -\frac{\lambda^2}{4\mu(t + t')} \right\}, \qquad \lambda \in \mathbf{R}^1$$

So, for the correlation function of the form given in Condition A, the only remaining unsolved case is $\alpha = 1$.

ACKNOWLEDGMENT

This research was supported, in part, by grants from the NSF, ONR and NRC.

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